V, W spazi vettoriali $\quad B_V$ base di V, B_W base di W

$f: V \to W \quad B'_V, \; B'_W$

$$[M(f)]^{B_V}_{B_W} \neq [M(f)]^{B'_V}_{B'_W}$$

$$B_V = \{v_1 \dots v_n\} \qquad B_W = \{w_1 \dots w_m\}$$

$$B'_V = \{b_1 \dots b_n\} \qquad B'_W = \{c_1 \dots c_m\}$$

$$v \in V \Rightarrow v = \lambda_1 v_1 + \dots + \lambda_n v_n = [v]_{B_V} = (\lambda_1 \dots \lambda_n)$$

$$b_j = b_{1j} \cdot v_1 + b_{2j} \cdot v_2 + \dots + b_{nj} v_n$$

$$B = (b_{ij})_{1 \leq ij \leq n} \in M_{n,n}(\mathbb{R}) \Rightarrow v = \mu_1(b_{11} v_1 + \dots b_{n1} v_n) + \dots$$
$$+ \mu_n(b_{1n} v_1 + \dots + b_{nn} v_n) =$$

$$\left([b_1]_{B_V} \; \dots \; [b_n]_{B_V} \right) = B$$

$$v \in V \Rightarrow \exists \mu_1 \dots \mu_n \text{ t.c. } v = \mu_1 b_1 + \dots + \mu_1 b_n \Leftrightarrow [v]_{B'_V}$$

$$(\mu_1 b_{11} + \mu_2 b_{12} + \dots + \mu b_n) v_1 + \dots + (\mu_1 b_{n1} + \dots + \mu_n b_{nn}) v_n$$

$$\Rightarrow [v]_{B_V} = \begin{pmatrix} \mu_1 b_{11} + \dots + \mu_n b_{1n} \\ \vdots \\ \mu_1 b_{n1} + \dots + \mu_n b_{nn} \end{pmatrix} = B \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = B[v]_{B'_V}$$

$$[w]_{B_W} = C[w]_{B'_W}$$

$$v \in V \Rightarrow f(v) \in W \qquad \Rightarrow C[f(v)]_{B'_W} = [M(f)]^{B_v}_{B_W} \cdot B[v]_{B_v}$$

$$\underset{\displaystyle C[f(v)]_{B'_W}}{\underset{\shortparallel}{[f(v)]_{B_W}}} = [M(f)]^{B_V}_{B_W} \cdot \underset{\displaystyle B \cdot [v]_{B'_V}}{\underset{\shortparallel}{[v]_{B_V}}} \Rightarrow [f(v)]_{B'_W} = \underset{\displaystyle [M(f)]^{B'_V}_{B'_W}}{\underset{\shortparallel}{\boxed{C^{-1} \cdot [M(f)]^{B_V}_{B_W} \cdot B}}} \cdot [v]_{B'_V}$$

Definizione

$M_1, M_2 \in M_{m,n}(\mathbb{R})$ si dicono equivalenti se

$\exists C \in M_{m,m}(\mathbb{R}) \; \exists B \in M_{n,n}(\mathbb{R})$ invertibili tale che $M_1 = C^{-1} M_2 B$

Proposizione

$f: V \to W$ omomorfismo

M_1 e M_2 rappresentano f rispetto a basi diverse

M_1 e M_2 sono simili

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "{[M(f)]^{B_v}_{B_W}}"] & & \mathbb{R}_m \\
V \arrow[u, "B_V", "\cong"'] \arrow[rr, "f"] \arrow[d, "\cong"', "B'_V"] & & W \arrow[u, "B_W"', "\cong"] \arrow[d, "\cong", "B'_W"'] \\
\mathbb{R}^n \arrow[uu, bend left, "B"] \arrow[rr, "{[M(f)]^{B_v}_{B_W}}"'] & & \mathbb{R}_m \arrow[uu, bend right, "C"']
\end{tikzcd}$$

Come funziona con gli endomorfismi?

$f: V \to V$ $\quad \underbrace{\mathcal{B}, \mathcal{B}'}_{\text{base}}$ basi di V $\quad \underbrace{B}_{\text{matrice}} =$ matrice di cambiamento di base

sono 2 cose diverse

$$\left[M(f)\right]^{\mathcal{B}}_{\mathcal{B}}, \quad \left[M(f)\right]^{\mathcal{B}'}_{\mathcal{B}}, \quad \text{ecc...}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "{[M(f)]^{B}_{B}}"] & \mathbb{R}_n \\
V \arrow[u, phantom, "\cong \mathcal{B}"] \arrow[r, "f"] \arrow[d, phantom, "\mathcal{B}' \cong"] & V \arrow[u, phantom, "\mathcal{B} \cong"] \arrow[d, phantom, "\cong \mathcal{B}'"] \\
\mathbb{R}^n \arrow[uu, bend left, "B"] \arrow[r, "{[M(f)]^{B}_{B}}"'] & \mathbb{R}_n \arrow[uu, bend right, "B"']
\end{tikzcd}$$

quindi $$\left[M(f)\right]^{\mathcal{B}'}_{\mathcal{B}'} = B^{-1}\left[M(f)\right]^{\mathcal{B}}_{\mathcal{B}} \cdot B$$

$$\left[M(f)\right]^{\mathcal{B}'}_{\mathcal{B}} = \left[M(f)\right]^{\mathcal{B}}_{\mathcal{B}} \cdot B$$

$$\left[M(f)\right]^{\mathcal{B}}_{\mathcal{B}'} = B^{-1}\left[M(f)\right]^{\mathcal{B}}_{\mathcal{B}}$$

Due matrici $M_1, M_2 \in M_{n,n}(\mathbb{R})$ sono simili se $\exists B \in M_{n,n}(\mathbb{R})$ invertibile tale che $M_1 = B^{-1} M_2 B$

Esercizi

$f: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x+2y+z \\ y+z \end{pmatrix}$$

1) f è lineare? Si perchè è dato da polinomi omogenei di primo grado

2) Scrivere $M(f)$ rispetto a $\mathbb{R}^3$ e $\mathbb{R}^2$

$$\mathcal{E}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{E}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M(f) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

3) $\operatorname{Ker} f \neq 0$?

$$\operatorname{Ker}(f) = \operatorname{Ker}(M(f))$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{cases} x - z = 0 \\ y + z = 0 \end{cases} \to \begin{cases} x = z \\ y = -z \end{cases} \Rightarrow \begin{pmatrix} z \\ -z \\ z \end{pmatrix}$$

$\operatorname{Ker} < \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} > \to$ è diverso da 0

$$\dim(\operatorname{Ker}(f)) = 1 = \operatorname{null}(f) \Rightarrow 2 = rk(A) = rk(f) = \dim(\operatorname{Im}(f))$$

2)

$$w = f\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix}$$

$$f^{-1}(w) = \left\{ v \in V \;\middle|\; f(v) = \begin{pmatrix}7\\9\end{pmatrix} \right\} = \begin{pmatrix}2\\1\\3\end{pmatrix} + \operatorname{Ker} f = \begin{pmatrix}2\\1\\3\end{pmatrix} + \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle$$

$$v = \begin{pmatrix}x\\y\\z\end{pmatrix} \qquad f(v) = \begin{pmatrix}x+2y+z\\y+z\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix}$$

$$\begin{cases} x + 2y + z = 7 \\ y + z = 9 \end{cases}$$

Esercizio 8 $T: \mathbb{R}^4 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 2x_1+x_3 \\ 2x_1+x_2+x_3 \\ x_2+2x_3 \end{pmatrix}$$

Scrivere $\left[M(f)\right]_{\mathcal{E}_3}^{\mathcal{E}_4}$

$$x_1\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[M(f)\right]_{\mathcal{E}_3}^{\mathcal{E}_4} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$\operatorname{Ker} f = \operatorname{Ker}(M(f))$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \xrightarrow{G-J} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{pmatrix} \to x_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\operatorname{Ker} f = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ $\quad \dim(\operatorname{Ker}(f)) = 1 = \operatorname{null}(f)$

$\operatorname{Im}(f) \left\langle \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \right\rangle$

↳ mettiamo a matrice e troviamo $rk = 3$ quindi è lin ind ciò vuol dire che è una base di f trovare $[M(t)]_{C}^{B}$

$t: R^4 \to R^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} 2x_1 + x_3 \\ 2x_1 + x_2 + x_3 \\ x_2 + 2x_3 \end{pmatrix}$$

relazione alla quale facciamo riferimento

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$C = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Poui risolvere usando queste formule

$$C^{-1} M(t)_{\mathcal{E}}^{\mathcal{E}} B$$

$$t(b_1) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad t(b_2) = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \quad t(b_3) = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \quad t(b_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[t(b_1)]_C = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad [t(b_2)]_C = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad [t(b_3)]_C = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix},$$

$$[t(b_4)]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$[M(t)]_{C}^{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$